Math 608 - Real Variables II
Definitions, Theorems and Propositions

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May 1, 2017

Contents

The propositions and theorems marked with a ⋆ indicate that the proof was important and relatively short, and thus should be learned for exams. Definitions are shown in **blue** while theorems are shown in **purple**.

1 Topology

1 (topology, open sets) Given a set X, $\tau \subseteq \mathcal{P}(X)$ is a topology if

1. $\emptyset, X \in \mathcal{T}$
2. $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$
3. $\{U_\alpha\}_{\alpha \in \mathcal{A}} \in \mathcal{T} \Rightarrow \cup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{T}$

The elements of the topology are called the open sets. A set C is called closed if C^C is open.

ex. If (X, d) is a metric space, then $U \subseteq X$ is open $\Leftrightarrow \forall x \in U$, $\exists r > 0$ such that $B(x, r) \subseteq U$.

We call $\mathcal{T} = \{\emptyset, X\}$ the indiscrete (or trivial) topology.

We call $\mathcal{T} = \mathcal{P}(X)$ the discrete topology.

If $Y \subseteq X$, the relative topology is $\mathcal{T}_Y = \{O \cap Y \mid O \in \mathcal{T}\}$.

2 (closure, interior, boundary) For an arbitrary $A \subseteq X$, let $\overline{A}$ denote the smallest closed set containing A, called the closure

$$\overline{A} = \bigcap \{C \subseteq X \mid A \subseteq C \text{ and } C \text{ is closed }\}.$$

We let the interior of A, A°, be the largest open set contained in A:

$$A^\circ = \bigcup \{O \subseteq X \mid O \subseteq A \text{ and } O \text{ is open }\}.$$

The boundary of A, δA is $\delta A = \overline{A} \backslash A^\circ$.

3 (limit point) We say p is a limit point (or accumulation point) of A provided for every open set $O \ni p$, $(O \cap A) \backslash \{p\} \neq \emptyset$. We let $A' = \text{acc}(A) = \{\text{limit points of } A\}$.

Proposition: $\overline{A} = A \cup \text{acc}(A)$. Thus, A is closed $\Leftrightarrow \text{acc}(A) \subseteq A \Leftrightarrow A = \overline{A}$.

4 (local base, base) For $x \in X$, a family $\mathcal{B}_x \subseteq \mathcal{T}$ is called a base for $\mathcal{T}$ at x (or a local base at x) provided

1. $\forall U \in \mathcal{B}_x, x \in U$

2. $\forall O \in \mathcal{T}$ such that $x \in O$, $\exists U \in \mathcal{B}_x$ such that $U \subseteq O$.

A base for $\mathcal{T}$ is a family $\mathcal{B} \subseteq \mathcal{T}$ such that for all $x \in X$, $\mathcal{B}_x = \{O \in \mathcal{B} \mid x \in O\}$ is a base for $\mathcal{T}$ at x.

$\mathcal{S} \subseteq \mathcal{T}$ is a subbase for the topology provided the set of all finite intersections of elements is a base.

Theorem: TFAE:

1. $\mathcal{B}$ is a base
2. every open set is the union of sets in $\mathcal{B}$
3. each $x \in X$ is contained in some $V \in \mathcal{B}$ and if $U, V \in \mathcal{B}$ and $x \in U \cap V$ then there exists some $W \in \mathcal{B}$ such that $x \in W \subseteq U \cap V$

Theorem: The topology generated by $\mathcal{E} \subseteq \mathcal{P}(X)$ contains $\emptyset, X$ and all unions of finite intersections of elements in $\mathcal{E}$.

5 (product topology) Suppose $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in \mathcal{A}}$ are topological spaces and let $X = \Pi_{\alpha \in \mathcal{A}} X_\alpha$. For $\alpha \in \mathcal{A}$, let π_α be the projection from X onto X_α, and set

$$\mathcal{S} = \{\pi_\alpha^{-1}(O) \mid O \in \mathcal{T}_\alpha, \alpha \in \mathcal{A}\}.$$

Then $\mathcal{S}$ is a subbase for the product topology.

Proposition: For $X = \Pi_{\alpha \in \mathcal{A}} X_\alpha$ and $x_n \in X$, then $x_n \to x$ in $X \Leftrightarrow \pi_\alpha(x_n) = x_n(\alpha) \to x(\alpha) = \pi_\alpha(x)$ for all $\alpha \in \mathcal{A}$.

6 (first/second countable) $(X, \mathcal{T})$ is first countable provided $\forall x \in X$, there exists a countable base at x.

$(X, \mathcal{T})$ is second countable provided there is a countable base for $\mathcal{T}$.

Proposition: If X is second countable, then X is separable. The converse is true in metric spaces.

7 (convergence in a topological space) For a topological space $(X, \mathcal{T})$ and a sequence $\{x_n\} \subseteq X$, $p \in X$ we say $\{x_n\}$ converges to p ($x_n \to p$) provided for every open $O \in \mathcal{T}$ with $p \in O$, $\exists N$ such that $\forall n \geq N$, $x_n \in O$.

Note: limits need not be unique.

Proposition: Suppose X is first countable, $A \subseteq X$, $p \in X$. Then $p \in \overline{A} \Leftrightarrow \exists$ sequence $\{x_n\}$ in A such that $x_n \to p$.

8 (cofinite and cocountable topologies) The cofinite topology is

$$\mathcal{T} = \{O \subseteq X \mid O^c \text{ is finite}\}.$$

The cocountable topology is

$$\mathcal{T} = \{O \subseteq X \mid O^c \text{ is countable}\}.$$

9 (T1/T2/T3/T4)

- T1 - $\forall x \in X$, $\{x\}$ is closed
- T2 (Hausdorff) - $\forall x \neq y$ in X, there exists $O_x, O_y \in \mathcal{T}$ such that $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \emptyset$
- T3 (regular) - $(X, \mathcal{T})$ is T1 and for all $x \in X$, closed C with $x \notin C$, there exists open $U, V \in \mathcal{T}$ such that $x \in U$, $C \subseteq V$, and $U \cap V = \emptyset$
- T4 (normal) - $(X, \mathcal{T})$ is T1 and for all disjoint closed sets A, B there exists open U, V with $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$

Fact: $T4 \Rightarrow T3 \Rightarrow T2 \Rightarrow T1$.

10 (continuous (at x)) Let $(X, \mathcal{T}), (Y, \sigma)$ be topological spaces. We say the function $f : X \to Y$ is continuous at $x \in X$ if for every open $O \in \sigma$ with $f(x) \in O$, there exists an open $U \in \mathcal{T}$ with $x \in U$ such that $f(U) \subseteq O$.

f is continuous if for every $x \in X$, f is continuous at x.

Proposition: TFAE for $f : X \to Y$

1. f is continuous
2. for every open O in Y, $f^{-1}(O)$ is open in X
3. for every closed C in Y, $f^{-1}(C)$ is closed in X
4. there is a subbase $\mathcal{S}$ for Y such that for every $O \in \mathcal{S}$, $f^{-1}(O)$ is open in X

11 (weak topology) For topological spaces $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in \mathcal{A}}$ and functions $f : X \to X_\alpha$ from a set X. Then $W((f_\alpha)_{\alpha \in \mathcal{A}})$ is the weakest (smallest) topology on X making each f continuous.

This topology is generated by sets of the form $f_\alpha^{-1}(U_\alpha)$ where $\alpha \in \mathcal{A}$ and U_α is open in X_α.x

12 (product space theorems) **Theorem 1:** If X_α is Hausdorff for each $\alpha \in \mathcal{A}$, then $X = \Pi_{\alpha \in \mathcal{A}} X_\alpha$ is Hausdorff.

Theorem 2: If X_α and Y are topological spaces and $X = \Pi X_\alpha$ then $f : Y \to X$ is continuous IFF $\pi_\alpha \circ f$ is continuous for each α.

Theorem 3: If X is a topological space, A is a nonempty set, and $\{f_n\}$ is a sequence in X^A then $f_n \to f$ in the product topology IFF $f_n \to f$ pointwise.

13 ($C(X)$) For a topological space $(X, \mathcal{T})$, let $C(X)$ be the set of all $\mathbb{R}$-valued continuous functions $f : X \to \mathbb{R}$.

Let $C_b(X) = BC(X)$ be all $\mathbb{R}$-valued bounded, continuous functions $f : X \to \mathbb{R}$. We equip $C_b(X)$ with the norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$.

Let $\ell_\infty(X)$ be the set of all $\mathbb{R}$-valued bounded functions.

Theorem: If X is normal then the topology on X is $W(C_b(X))$.

14 (Urysohn's Lemma) Let $(X, \mathcal{T})$ be normal. If A, B are disjoint closed sets and $a \neq b$ in $\mathbb{R}$. Then there exists some $f \in C(X, [a, b])$ such that $f|_A \equiv a$ and $f|_B \equiv b$.

proof uses nastay lemma

15 (Tiktze Theorem) **Version 1:** Let $(X, \mathcal{T})$ be normal. If $A \subseteq X$ is closed and $f \in C(A, (a, b))$ then there exists some $F \in C(X, [a, b])$ such that $F|_A = f$.

Version 2: Let $(X, \mathcal{T})$ be normal. If $A \subseteq X$ is closed and $f \in C(A, (a, b))$ then there exists some $F \in C(X, \mathbb{R})$ such that $F|_A = f$.

X is called completely regular (or a $T_{3\frac{1}{2}}$ space) if X is T_1 and for each closed $A \subseteq X$, $x \notin A$ there exists some $f \in C(X, [0, 1])$ such that $f(x) = 1$, $f = 0$ on A.

2 Nets

16 (net) $(D, \leq)$ is called a directed set if

- $a \leq a$
- if $a \leq b$ and $b \leq c$ then $a \leq c$
- $\forall \alpha, \beta \in D$, $\exists \gamma \in D$ such that $\alpha \leq \gamma$, $\beta \leq \gamma$

A net in X is a function from a directed set into X.

For $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ a net and $\alpha_0 \in \mathcal{A}$, the tail of the net is $T_{\alpha_0} = \{x_\alpha \mid \alpha \in \mathcal{A}, \alpha \geq \alpha_0\}$.

17 (further net definitions) We say a net $\{x_\alpha\}$ is frequently in a set C if $T_\alpha \cap C \neq \emptyset$ for all $\alpha \in \mathcal{A}$.

We say a net $\{x_\alpha\}$ is eventually in a set C if there exists some $\alpha_0 \in \mathcal{A}$ such that $T_{\alpha_0} \subseteq C$.

(Note: eventually in $C \Rightarrow$ frequently in C)

Suppose $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ is a net in the topological space X, and $p \in X$. Then $x_\alpha \to p$ means for every open set O with $p \in O$, $\{x_\alpha\}$ is eventualy in O.

We say p is a cluster point of the net if for every open set $O \ni p$ the net $\{x_\alpha\}$ is frequently in O.

We say $\{y_\beta\}_{\beta \in \mathcal{B}}$ is a subnet of $\{x_\alpha\}$ provided there exists some $h : \mathcal{B} \to \mathcal{A}$ such that

- $\forall \alpha_0 \in \mathcal{A}$, $\exists \beta_0 \in \mathcal{B}$ such that for all $\beta \geq \beta_0$, $h(\beta) \geq \alpha_0$
- $x_{h(\beta)} = y_\beta$ for all $\beta \in \mathcal{B}$

ex. For $1, 2, 3, 4, \dots$ then $2, 1, 3, 2, 4, 3, 5, 4, \dots$ is a subnet but not a subsequence

Theorem: For a net $\{x_\alpha\}$ in $(X, \mathcal{T})$ TFAE:

1. x is a cluster point of $\{x_\alpha\}$
2. there exists a subnet $\{y_\beta\}$ of $\{x_\alpha\}$ such that $y_\beta \to x$

Theorem: f is continuous at $x \Leftrightarrow$ for all nets $x_\alpha \to x$, $f(x_\alpha) \to f(x)$

Theorem: For $D \subseteq X$, $p \in \overline{D} \Leftrightarrow$ there exists a net x_α in D s.t. $x_\alpha \to x$.

3 Compactness

18 (notions of compactness)

1. A is compact i.e. every open cover has a finite subcover

   1'. every family of closed sets with the finite intersection property has a non-empty intersection
2. X is sequentially compact if every sequence has a convergent subsequence.
3. X is countable compact if every countable open cover has a finite subcover.

   3'. If $C_1 \supseteq C_2 \supseteq \dots$ are closed and non-empty then $\cap C_n \neq \emptyset$
4. every infinite subset of X has a limit point

1\. $\Leftrightarrow 1'. \Rightarrow 3.$ and $2. \Rightarrow 3. \Leftrightarrow 3'. \Rightarrow 4.$

Theorem: If X is compact and $C \subseteq X$ is closed, then C is compact.

Theorem: If X is Hausdorff, then compact sets are also closed.

Theorem: If $f : X \to Y$ is continuous and $C \subseteq X$ is compact then $f(C)$ is compact.

19 (net compactness) We call a net $\{x_\alpha\}$ universal if for all $Y \subseteq X$, if the net is frequently in Y then the net is eventually in Y.

Lemma: every net has a universal subnet.

Theorem: For a topological space $(X, \mathcal{T})$, TFAE:

1. X is compact
2. every net in X has a cluster point
3. every net in X has a convergent subnet
4. every universal net in X converges

20 (locally compact) A topological space is called locally compact if every point has a compact neighborhood. Locally compact Hausdorff spaces are abbreviated LCH.

Equivalently, every point has an open neighborhood U with closure $\overline{U}$ compact.

21 (Tychonoff Theorem) If (X_α) are compact topological spaces, then $X = \Pi_{\alpha \in \mathcal{A}} X_\alpha$ (with the product topology) is compact.

Theorem: Axiom of Choice $\Leftrightarrow$ Tychonoff

22 (equicontinuous) Let $(X, \mathcal{T})$ be a topological space, $\mathcal{J} \subseteq C(X, (Z, \|\cdot\|))$. For $x \in X$ we say $\mathcal{J}$ is equicontinuous at x provided for all $\epsilon > 0$ there exists some neighborhood U_x of x such that

$$\sup_{f \in \mathcal{J}} \sup_{y \in U_x} \|f(x) - f(y)\| \leq \epsilon.$$

We say $\mathcal{J}$ is equicontinuous if it is equicontinuous at x for all $x \in X$.

We say $\mathcal{J}$ is pointwise bounded if for all $x \in X$, $\sup_{f \in \mathcal{J}} \|f(x)\| < \infty$.

23 (Arzela-Ascoli) We say a metric space X is totally bounded if for any $r > 0$, X can be covered by a finite number of balls of radius r.

Arzela-Ascoli Let X be a compact Hausdorff space. If $\mathcal{F}$ is an equicontinuous, pointwise bounded subset of $\mathcal{C}(X)$ then $\mathcal{F}$ is totally bounded in the uniform metric and the closure of $\mathcal{F}$ in $\mathcal{C}(X)$ is compact.

Alternative version 1: Let X be a σ-compact LCH space. If $\{f_n\}$ is an equicontinuous, pointwise bounded sequence in $\mathcal{C}(X)$, then there exists a $f \in \mathcal{C}(X)$ and a subsequence of $\{f_n\}$ that converges to f uniformly on compact sets.

Alternative version 2: Let X be compact and $\mathcal{F} \subseteq \mathcal{C}(X)$. Then $\overline{\mathcal{F}}$ is compact in $\mathcal{C}(X)$ IFF

1. $\mathcal{F}$ is equicontinuous
2. $\mathcal{F}$ is pointwise bounded

24 (Stone-Weierstrass) $\mathcal{A}$ is called an algebra if it is a real vector subspace of $C(X)$

such that $fg \in \mathcal{A}$ whenever $f, g \in \mathcal{A}$.

Let X be a compact, Hausdorff space and $\mathcal{B} \subseteq \mathcal{C}(X, \mathbb{R})$ a subalgebra such that $\mathcal{B}$ separates points (that is, for $x \neq y, \exists f \in \mathcal{B}$ with $f(x) \neq f(y)$). Then if there exists some $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathcal{B}$, then $\overline{\mathcal{B}} = \{f \in \mathcal{C}(X, \mathbb{R}) \mid f(x_0) = 0\}$. Otherwise, $\overline{B} = \mathcal{C}(X)$.

4 Normed Spaces

25 (complete) A Banach space is a complete normed vector space. It is called complete if every Cauchy sequence converges in X.

Theorem: X is complete $\Leftrightarrow$ when $\sum_{n \in \mathbb{N}} \|x_n\| < \infty$ then $(\sum_{n=1}^{N} x_n)_N$ converges in X

26 (linear equivalences) If X, Y are normed spaces and $T : X \to Y$ is linear, then TFAE:

1. T is continuous
2. T is continuous at 0
3. T is bounded

   that is, there exists some $c > 0$ such that $\|Tx\| \leq c\|x\|$.

   Equivalently, $\sup_{\|x\| \leq 1} \|Tx\| < \infty$
4. T is Lipschitz

   that is, there exists some c such that $\|Tx - Ty\| \leq c\|x - y\|$

We denote $L(X, Y) = \{T : X \to Y \mid T \text{ is continuous and linear}\}$ with norm $\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$. Let $X^* = L(X, \mathbb{R})$ be the dual of X.

27 (invertibility) Suppose X is a Banach algebra with identity, $\|I\| = 1$. Then

if $\|I - a\| < 1$ then a is invertible and $\|a^{-1}\| \leq \frac{1}{1 - \|I - a\|}$

if y is invertible and $\|y - x\| < \frac{1}{\|y^{-1}\|}$ then x is invertible (so invertible elements are open)

5 Quotient Spaces

28 (algebra quotient) If X is a normed space and $M \subseteq X$ is a closed subspace, then $X/M = \{x + M \mid x \in X\}$ is the algebra quotient with $(x + M) + (y + M) = (x + y) + M$. We have the linear surjection

$$\begin{aligned} \pi_M : X &\to X/M \\ x &\mapsto x + M \end{aligned}$$

where $\ker(\pi_M) = M$. Put the norm on X/M to be

$$\|x + M\| = \inf\{\|y\| \mid y \in x + M\} = \inf\{\|x - m\| \mid m \in M\} = \text{dist}(x, M)$$

If M is not closed, this is merely a seminorm. Then $\pi_M(B_X(1, 0)) = B_{X/M}(0, 1)$ so π_M is continuous and $\|\pi_M\| = 1$.

29 (Hahn-Banach) For a real vector space X, we say $p : X \to \mathbb{R}$ is a sublinear mapping if $p(x + y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ when $\lambda \geq 0$.

Hahn-Banach: Let X be a real vector space, p a sublinear functional on X, M a subspace of X, and f a linear functional on M such that $f|_M \leq p|_M$. Then there exists a linear functional F on X such that $F \leq p$ on X and $F|_M = f$.

For the complex case, we require $|f(x)| \leq p(x)$ and we get $|F(x)| \leq p(x)$.

30 (Applications of Hahn-Banach)

1. If M is a closed subspace of X and $x \in X\ M$ then there exists $f \in X^*$ such that $f(x) \neq 0$ and $f|_M = 0$. In fact, if $\delta = \inf_{y \in M} \|x - y\|$, f can be taken to satisfy $\|f\| = 1$ and $f(x) = \delta$.
2. If $x \neq 0 \in X$, there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$
3. The bounded linear functionals on X separate points

31 (reflexive) **Theorem:** If $x \in X$, define $\hat{x} : X^* \to \mathbb{C}$ by $\hat{x}(f) = f(x)$. Then the map $x \mapsto \hat{x}$ is a linear isometry from X into X^{**}.

We call X reflexive if $X^{**} = X$. Equivalently, X is reflexive if ˆ is surjective.

Theorem: Suppose X is a Banach space and M is a closed subspace. Then

1. X is reflexive $\Rightarrow$ M is reflexive
2. X reflexive $\Leftrightarrow$ X^* reflexive
3. X reflexive $\Rightarrow$ X/M reflexive
4. X reflexive $\Leftrightarrow$ $\overline{B_X(0, 1)}$ is weakly compact

32 (Baire Category) We say C is nowhere dense if $(\overline{C})^\circ = \emptyset$.

Theorem: Let X be a complete metric space. Then if $\{U_n\}$ is a sequence of open dense sets, $\cap U_n$ is dense. Thus, X is not a countable union of nowhere dense sets.

A set that is a countable union of nowhere dense sets is said to be of first category (and it's complement is called residual). A set which is not a countable union of nowhere dense sets is called second category.

33 (uniform boundedness principle) Let X be a Banach space and Y a normed space, $\mathcal{S} \subseteq L(X,Y)$ where $\mathcal{S}$ is pointwise bounded (i.e. $\forall x \in X, \sup\{\|Tx\| \mid T \in \mathcal{S}\} < \infty$).

Then $\mathcal{S}$ is uniformly bounded (i.e. $\sup_{T\in\mathcal{S}} \|T\| < \infty$.

34 (Banach-Steinhaus) Suppose X is a Banach space and Y is a normed space, and $\{T_n\} \subseteq L(X,Y)$ and for all $x \in X$, $T_n x \to Tx$ in Y. Then $T \in L(X,Y)$.

35 (open mapping theorem) **little open mapping theorem:** Suppose X is a Banach space and Y is a normed space, $T \in L(X,Y)$ and $r > 0$. Then if $\overline{T(B(0,1))} \supseteq B(0,r)$ then $T(B(0,1)) \supseteq B(0,r)$.

open mapping theorem: Suppose X, Y are Banach spaces and $T \in L(X,Y)$ is surjective. Then T is an open mapping.

Remark: For a linear map T, T is open $\Leftrightarrow \exists r > 0$ such that $T(B(0,1)) \supseteq B(0,r)$.

36 (closed graph) For Banach spaces X, Y and $T : X \to Y$ linear, then $T \subseteq X \times Y$ is closed $\Leftrightarrow$ T is a bounded linear operator.

6 Topological Vector Spaces

37 (TVS) Let X be a vector space, $\mathcal{T}$ a topology on X. Then $(X,\mathcal{T})$ is a TVS provided

- $+ : X \times X \to X$ is continuous
- $\cdot : \mathbb{R} \times X \to X$ is continuous

ex. normed spaces under the weak topology $w(X^*)$.

We say the TVS $(X,\mathcal{T})$ is locally convex there exists a local base for $\mathcal{T}$ consisting of convex sets.

Theorem: If $(X,\mathcal{T})$ is a locally convex TVS, then there exists seminorms $\{p_\alpha \mid \alpha \in \mathcal{A}\}$ such that $\mathcal{T} = w(p_\alpha)$.

38 (gauge function) Define the gauge function (or Minkowski) of a convex set U in the vector space X to be

$$P_U(x) = \inf\{\lambda > 0 \mid \frac{x}{\lambda} \in U\}$$

We say p is an internal point of U (that is, $\forall y \in X, \exists \epsilon > 0$ such that $p + [|z| \leq \epsilon]y \subseteq U$). If 0 is an interval point, then the gauge function is defined since the set is non-empty. Then

1. $P_U(\lambda x) = \lambda P_U(x)$ if $\lambda \geq 0$
2. $P_U(x+y) \leq P_U(x) + P_U(y)$
3. If U is balanced, then $P_U(\lambda x) = |\lambda| P_U(x)$

39 (Separation Theorem / Geometric Hahn-Banach) Say X is a LCTVS over $\mathbb{R}$ and $U, C \subseteq X$ are convex sets such that $U \cap C = \emptyset$ and $U^\circ \neq \emptyset$. Then there exists some non-zero $f \in X^*$ and some $\alpha \in \mathbb{R}$ such that $U \subseteq [f < \alpha]$ and $C \subseteq [f \geq \alpha]$

Corollary 1: If $(X, \mathcal{T})$ is Hausdorff LCTVS, then X^* separates points of X

Corollary 2: If $(X, \mathcal{T})$ is a LCTVS, $C \subseteq X$ is convex, then $\overline{C}^{\text{weak}} = \overline{C}^{\mathcal{T}}$.

Corollary 3: If X is a normed space and $A \subseteq X$, then A is norm bounded $\Leftrightarrow$ A is weakly bounded (where A is weakly bounded if for all $x^* \in X^*$, $\sup_{x \in X} |\langle x^*, x \rangle| < \infty$

40 (topologies on X^*) On X^* we have three topologies:

weak$^*(w(\hat{X})) \subseteq$ weak$(w(X^{**})) \subseteq$ norm topology

Where the first is the topology of pointwise convergence on X and the first $\subseteq$ is equality if X is reflexive.

41 (Banach-Alaoglu) If X is a normed space, then $\overline{B_{X^*}} = \{x^* \in X^* \mid \|x^*\| \leq 1\}$ is weak*-compact.

Corollary: If X is reflexive, then $\overline{B_{X^*}}$ is weakly compact.

X is reflexive if and only if $\overline{B_X}$ is weakly compact.

42 (Goldstine) Suppose X is normed. Then $\widehat{B_X}$ is weak*-dense in $B_{X^{**}}$, $\widehat{B_X} \subseteq B_{X^{**}}$ where we have equality IFF X is reflexive.

43 (random theorems) **Theorem 1:** Suppose X is a normed space.

1. There exists a compact Hausdorff space K such that X is isometrically isomorphic to a subspace of $C(X)$.
2. If X is separable, K can be taken to be a compact metric space

   Moreover, if X is separable, K can be taken to be the Cantor set $\{0,1\}^{\mathbb{N}}$ or $[0,1]$.

Theorem 2: If X is a normed space, (B_{X^*}, weak^*) is metrizable $\Leftrightarrow$ X is separable.

44 (completely regular) $\mathcal{F} \subseteq C(X)$ is said to separate points from closed sets proved for all $x \in X$ and closed set $C \subseteq X$ with $x \notin C$, then there exists some $f \in \mathcal{F}$ such that $f(x) \notin \overline{f(C)}$.

If $C(X)$ separates points from closed sets, X is said to be completely regular.

Proposition If $(X, \mathcal{T})$ is completely regular, then $\mathcal{T} = w(C(X)) = w(C(X, [0, 1]))$.

7 L^p spaces

45 ($L^p(\mu)$) For $0 < p < \infty$, let $L^p = \{$ real-valued measurable functions $f \mid \int |f|^p d\mu < \infty\}$ with $\|f\|_p = \left(\int |f|^p d\mu\right)^{1/p}$. This is a norm for $1 \leq p < \infty$.

Let $L^\infty = \{$ all bounded measurable functions $\}$ with supremum norm defined by

$$\|f\|_\infty := \inf\left\{\sup_{x \in E^C} |f(x)| \mid \mu(E) = 0\right\} = \inf_{h \in L^\infty, h=0 a.e.} \|f - h\|_{\sup}$$

46 (Riesz-Fisher) For $1 \leq p < \infty$, L^p is complete

47 (Hölder's inequality) Let q be the conjugate exponent of p so $\frac{1}{p} + \frac{1}{q} = 1$ (i.e. $q = \frac{p}{p-1}$)

For measurable f, g and $1 < p < \infty$ then $\|fg\|_1 \leq \|f\|_p \|g\|_q$.

If $f \in L^p$ and $g \in L^q$ if and only if $f = 0$ a.e. OR $g = 0$ a.e. OR $|f|^p$ is a scalar multiple of $|g|^q$.

If $f \in L^p$ then $\|f\|_p = \max\left\{\int fg d\mu \mid \|g\|_q \leq 1\right\}$ (maximum is achieved! by $g = \text{sgn}(f)$).

Alternate Hölder's inequality: For $0 < \lambda < 1$, then $\int |f|^\lambda |g|^{1-\lambda} \leq \left(\int |f|\right)^\lambda \left(\int |g|\right)^{1-\lambda}$.

48 (Minkowski) For $1 \leq p < \infty$, $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

49 (simple functions in L^p) Let Σ be all measurable simple functions supported on sets of finite measure. Then Σ is dense in L^p for $0 < p < \infty$.

Continuation of Hölder: Assume μ is σ-finite, $1 < q < \infty$ and g is measurable. Then $\|g\|_q = \sup\{\int fg \mid \|f\|_p \leq 1, f \in \Sigma\}$.

If μ is semifinite (i.e. if $\mu(A) = \infty$ then $\exists B \subseteq A$ such that $0 < \mu(B) < \infty$) then $\|g\|_\infty = \sup\{\int fg \mid \|f\|_1 \leq 1, f \in \Sigma\}$.

50 (dual of L^p) For $1 \leq p < \infty$ and q the conjugate exponent of p, then there is a mapping $J_p : L^q \to (\mathbb{R})^{L^p}$ defined via $J_p(g)(f) = \int fg \leq \|g\|_q \|f\|_p$.

So $J_p(g)$ is linear and $\|J_p(g)\|_{(L^p)^*} \leq \|g\|_q$.

Moreover, J_p is linear, so J_p is a linear operator which maps L^q onto $(L^p)^*$ and $\|J_p\| \leq 1$.

Theorem: J_p is surjective if $1 < p < \infty$ and J_1 is surjective if μ is σ-finite.

Corollary: For $1 < p < \infty$, then L^p is reflexive.

51 (relations between L^p as p varies) **Theorem 1:** If $\mu(X) < \infty$ and $0 < p < r \leq \infty$ then if f is measurable, $\|f\|_p \leq \mu(X)^{\frac{1}{p}-\frac{1}{r}}\|f\|_r$

Theorem 2: If μ is the counting measure, then for $0 < p < r \leq \infty$, $\|f\|_p \geq \|f\|_r$. Therefore, $\ell^p \subseteq \ell^r$.

Theorem 3: If $0 < p < q < r \leq \infty$ and $\frac{1}{q} = \lambda\frac{1}{p} + (1-\lambda)\frac{1}{r}$ then

$$\|f\|_q \leq \|f\|_p^{\lambda}\|f\|_r^{1-\lambda}$$

52 (inequalities) **Chebychev's inequality:** For $f \in L^p$ for $0 < p < \infty$ and any $\alpha > 0$,

$$\alpha^p\mu[|f| > \alpha] \leq \|f\|_p^p$$

Theorem: For σ-finite measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ and $K : X \times Y \to \mathbb{R}$ define

$$(T_K f)(y) = \int_X K(x,y)f(x)d\mu(x)$$

Suppose there exists c such that $\int_X |K(x,y)|d\mu(x) \leq c$ and $\int_Y |K(x,y)|d\nu(y) \leq c$ and K is $\mu \times \nu$ measurable. Then T_K maps $L^p(\mu)$ into $L^q(\nu)$ for all $1 \leq p \leq \infty$, and $\|T_K\|_{L^p(\mu)\to L^q(\nu)} \leq c$.

53 (Distribution functions) Define the total distribution function of $f \in L^0$ via

$$\lambda_f : [0,\infty) \to [0,\infty]$$
$$t \mapsto \mu[|f| > t]$$

Proposition:

1. λ_f is decreasing and right continuous
2. $|f| \leq |g|$ implies $\lambda_f \leq \lambda_g$
3. If $|f_n| \uparrow |f|$ a.e. then $\lambda_{f_n} \uparrow \lambda_f$
4. $\lambda_{g+h}(t) \leq \lambda_g(t/2) + \lambda_h(t/2)$

Theorem: Take $\phi : (0,\infty) \to \mathbb{R}^+$ Borel measurable and $\lambda_f(t) < \infty$ for all $t > 0$. Then

$\int_X \phi \circ |f| d\mu = -\int_0^\infty \phi(t) d\lambda_f(t)$.

$$\|f\|_p^p = -\int_0^\infty t^p d\lambda_f(t) = p\int_0^\infty t^{p-1}\mu[|f| > t]dt$$

54 (weak $L^p = L^{p,\infty}$) Define $[f]_p^p = \sup_{t>0} t^p \lambda_f(t)$. Let $L^{p,\infty} = \{f \in L^0 \mid [f]_p^p < \infty\}$, called the weak L^p.

$[cf]_p = |c|[f]_p$

$[f+g]_p^p \le 2^p([f]_p^p + [g]_p^p)$.

8 Abstract Interpolation Theory

55 (compatible couple / intermediate space) We call a pair on Banach spaces $\widetilde{X} = (X_0, X_1)$ a compatible couple (or interpolation pair) if there exists a TVS Z such that $X_0 \cup X_1 \subseteq Z$ and the inclusion mappings $X_0 \to Z$, $X_1 \to Z$ are both continuous linear operators. WLOG: $X_0 + X_1 = Z$, where

$$\|z\| := \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} \mid x_0 \in X_0, x_1 \in X_1 \text{ such that } z = x_0 + x_1\}$$

Denote $Z = \Sigma(\widetilde{X})$, define $\Delta(\widetilde{X}) = X_0 \cap X_1 \subseteq \Sigma(\widetilde{X})$ with norm $\|z\| = \max\{\|z\|_{X_0}, \|z\|_{X_1}\}$.

If $\Delta(\widetilde{X}) \subseteq X \subseteq \Sigma(\widetilde{X})$ we say X is an intermediate space to X_0 and X_1.

56 (interpolation pair) Suppose $T : \Sigma(\widetilde{X}) \to \Sigma(\widetilde{Y})$ is a bounded linear operator. We write $T \in L(\widetilde{X}, \widetilde{Y})$ provided $T(X_0) \subseteq Y_0$ and $T(X_1) \subseteq Y_1$ and

$$\|T|_{X_0}\|_{X_0 \to Y_0} \vee \|T|_{X_1}\|_{X_1 \to Y_1} < \infty.$$

We say that (X, Y) is an interpolation pair for $\widetilde{X}, \widetilde{Y}$ if

1. X is an intermediate space for X_0 and X_1
2. Y is an intermediate space for Y_0 and Y_1
3. whenever $T \in L(\widetilde{X}, \widetilde{Y})$ then $T|_X \in L(X, Y)$

We say (X, Y) is an exact interpolation pair for $\widetilde{X}, \widetilde{Y}$ if

$$\|T|_X\|_{X \to Y} \le \|T|_{X_0}\|_{X_0 \to Y_0} \vee \|T|_{X_1}\|_{X_1 \to Y_1}$$

If $0 < t < 1$ then we say (X, Y) is an interpolation pair for $\widetilde{X}$, $\widetilde{Y}$ of exponent t provided there exists some $C < \infty$ such that for all $T \in L(\widetilde{X}, \widetilde{Y})$,

$$\|T|_X\|_{X\to Y} \le C\, \|T|_{X_0}\|_{X_0\to Y_0}^{1-t}\, \|T|_{X_1}\|_{X_1\to Y_1}^{t}$$

Then we can say (X, Y) is an exact interpolation pair for $\widetilde{X}, \widetilde{Y}$ provided $C = 1$.

57 **($\mathcal{H}(\widetilde{X})$)** For complex Banach space $\widetilde{X} = (X_0, X_1)$ let

$$\mathcal{H}(\widetilde{X}) = \left\{ f : S \to \Sigma(\widetilde{X}) \mid \substack{f \text{ is continuous, bounded, and analytic on } \int(S)=S^0 \\ f(is)\in X_0,\ f(1+is)\in X_1 \\ \sup_{s\in\mathbb{R}} \|f(is)\|_0<\infty,\ \sup_{s\in\mathbb{R}} \|f(1+is)\|_1<\infty} \right\}$$

where $S = \{z \in \mathbb{C} \mid 0 \le \Re(z) \le 1\}$. Define

$$\|f\|_{\mathcal{H}(\widetilde{X})} := \sup_{s\in S} \|f(is)\|_0 \vee \|f(1+is)\|_1$$

Then for any $z \in S$,

$$\|f(z)\|_{\Sigma(\widetilde{X})} \le \sup_{s\in S} \|f(is)\|_{\Sigma(\widetilde{X})} \vee \|f(1+is)\|_{\Sigma(\widetilde{X})} \le \|f\|_{\mathcal{H}(\widetilde{X})}$$

For $0 < t < 1$, define

$$X_t = \{x \in \Sigma(\widetilde{X}) \mid \exists f \in \mathcal{H}(\widetilde{X}) \text{ with } f(t) = x\} \qquad \|x\|_t = \inf\{\|f\|_{\mathcal{H}(\widetilde{X})} \mid f(t) = x\}$$

So $X_t = \mathcal{H}(\widetilde{X})/N_t(\widetilde{X})$ where $N_t(\widetilde{X}) = \{f \in \mathcal{H}(\widetilde{X}) \mid f(t) = 0\}$.

Theroem: $\widetilde{X} = (X_0, X_1)$ and $\widetilde{Y} = (Y_0, Y_1)$ compatible, then for $0 < t < 1$, (X_t, Y_t) is an exact interpolation space of exponent t between $\widetilde{X}$ and $\widetilde{Y}$.

Theorem: Let $X_0 = L^{p_0}(\mu)$, $X_1 = L^{p_1}(\mu)$ for $1 \le p_0, p_1 \le \infty$ be complex spaces. Let $\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}$. Then $L^{p_t}(\mu) = X_t$ with equality of norms.

Note: For real L^p we get the same theorem, except exactness must be removed.

58 **(Riesz-Thorin)** If $1 \le p_0, p_1 \le \infty$, $1 \le q_0, q_1 \le \infty$ and $0 < t < 1$ with

$$\frac{1}{p_t} := \frac{t}{p_0} + \frac{1-t}{p_1} \qquad \frac{1}{q_t} := \frac{t}{q_0} + \frac{1-t}{q_1}$$

Suppose $X_0 = L^{p_0}(\mu)$, $X_0 = L^{p_1}(\mu)$ and $Y_0 = L^{q_0}(\nu)$, $Y_1 = L^{q_1}(\nu)$ (compatible couple).

Then for $0 < t < 1$, $L^{p_t}(\mu) + L^{q_t}(\nu)$ is an exact interpolation pair for $\widetilde{X} = (X_0, X_1), \widetilde{Y} = (Y_0, Y_1)$.

59 (Marcinkiewicz Interpolation) Let $(X, \mathcal{M}, \mu)$ be a measure space and D a subspace of $L^0(\mu)$. We say $T : D \to L^0(\nu)$ is sublinear if

1. $|T(f+g)| \leq |Tf| + |Tg|$
2. $|T(cf)| = c|Tf|$ if $c \geq 0$

T is said to be of strong type (p, q) if $T(L^p(\mu)) \subseteq L^q(\nu)$ and $\|T|_{L^p(\mu)}\|_{L^p(\mu)\to L^q(\nu)} < \infty$.

T is said to be of weak type (p, q) if $T(L^p(\mu)) \subseteq L^{q,\infty}(\nu)$ and $\|T|_{L^p(\mu)}\|_{L^p(\mu)\to L^{q,\infty}(\nu)} =: \sup_{\|x\|_{L^p(\mu)}\leq 1}[Tx]_{q,\infty} < \infty$ where for $q < \infty$, $L^{q,\infty}(\nu) = \{f \in L^0(\nu) \mid \sup_t t^{1/q}\nu[|f| > t] =: [f]_{q,\infty} < \infty\}$.

Weak type (p, ∞) is the same as strong type (p, ∞).

Marcinkiewicz Interpolation Theorem: $1 \leq p_0 \leq q_0 \leq \infty$ and $1 \leq p_1 \leq q_1 \leq \infty$, $q_0 \neq q_1$ and $0 < t < 1$,

$$\frac{1}{p_t} := \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \frac{1}{q_t} := \frac{1-t}{q_0} + \frac{t}{q_1}$$

If $T : L^{p_0}(\mu) + L^{p_1}(\mu) \to L^0(\nu)$ is sublinear, and is of weak type $(p_0, q_0$ and weak type (p_1, q_1) then T is of strong type (p_t, q_t) for all $0 < t < 1$ and

$$\|T\|_{L^{p_t}\to L^{q_t}} \leq \frac{C\Big(\|T\|_{L^{p_0}\to L^{p_0,\infty}} \vee \|T\|_{L^{p_1}\to L^{p_1,\infty}}\Big)}{t(1-t)}$$

where $C = C(p_0, p_1, q_0, q_1)$ is some constant $< \infty$.

9 Baire σ-algebra

60 (Baire σ-algebra) Define $\mathcal{B}a(X) = \sigma\{[f > a] \mid a \in \mathbb{R}, f \in C(x)\} = \sigma\{[f > 0] \mid f \in C(X)\}$. If X is metrizable, then the Baire set is equal to the Borel set.

Lemma: If X is normal, then

$$\mathcal{B}a(X) = \sigma\{\text{ open } F_\sigma \text{ set }\} = \sigma\{\text{ closed } G_\delta \text{ sets }\} = \sigma\{E \mid \ E \text{ is both } F_\sigma \text{ and } G_\delta\}$$

Let $\mathcal{M}(X)$ be all finite Baire signed measures on X. We have a norm $\|\mu\|_{var} = |\mu|(X) = \mu^+(X) + \mu^-(X)$.

Define $J : \mathcal{M}(X) \to C(X)^*$ by $J(\mu)(f) = \int_X f(x) d\mu(x)$. Then J is a linear mapping into $C(X)^*$. In fact, J is an isometric isomorphism and is surjective.

61 (Boolean) Suppose (X, τ) is compact Hausdorff. TFAE:

1. X is Boolean (i.e. X has a base of clopen sets
2. The continuous simple functions are dense in $C(X)$

   $\mathcal{C}l(X) = \{E \subseteq X \mid E \text{ is clopen }\}$

   $\mathcal{S}(X) =$ continuous simple functions $= \text{span}\{\chi_E \mid E \in \mathcal{C}l(X)\}$.
3. $\forall x \neq y$, there exists a clopen U such that $x \in U, y \in U^C$
4. X is homeomorphic to a closed subset of $\{0,1\}^B$ for some B
5. X is totally disconnected

10 Regularity Properties of Measures

62 (regular) A measure μ on X is innerregular if $\forall E \in \mathcal{M}$,

$$\mu(E) = \sup\{\mu(K) \mid K \subseteq E \text{ compact } \ K \in \mathcal{M}\}$$

A measure μ on X is outerregular if $\forall E \in \mathcal{M}$,

$$\mu(E) = \inf\{\mu(U) \mid E \subseteq U \text{ open } \ U \in \mathcal{M}\}$$

We say μ is regular if it is both inner and outer regular.

If μ is a finite signed measure, we say μ is regular if both μ^+ and μ^- are regular ($\Leftrightarrow |\mu|$ is regular).

Theorem: If X is compact and Hausdorff, then if μ is a finite Baire measure, μ is regular.

Corollary: Suppose $(X, \mathcal{T})$ is a LCTVS and $K \subseteq X$ is weakly compact, $x_0 \in \overline{\text{conv}(K)}$. Then x_0 is the Baray center of a Baire probability measure on K. That is, $\forall x^* \in X^*$, $\langle x^* x_0 \rangle = \int_K \langle x^*, x \rangle d\mu(x)$.

63 (dual of $C(X)$) If X is compact and Hausdorff, then

$$C(X)^* = \{ \text{ finite signed regular Borel measures } \}$$

64 (Krein-Milman) If C is a convex set in a real vector space, then $x \in C$ is said to be an extreme point provided whenever $y, z \in C$ and $0 < \lambda < 1$, $x = \lambda y + (1-\lambda)z$ then $x = y = z$.

Krein-Milman Lemma: If X is a Hausdorff LCTVS, and $C \subseteq X$ is a non-empty, compact, convex set then $\text{ext}(C) \neq \emptyset$.

Krein-Milman Theorem: If X is a Hausdorff LCTVS, $C \subseteq X$ is a non-empty, compact, convex set, then $C = \overline{\text{conv}(\text{ext}(C))}$, where $\text{ext}(C) = \{$ all extreme points of C $\}$.

Remark 1: If X is a reflexive Banach spac, then B_X is weakly compact, hence $B_X = \overline{\text{conv}(\text{ext}(B_X))}^{\text{weak}} = \overline{\text{conv}(\text{ext}(B_X))}^{\text{norm}}$.

Remark 2: Suppose X is a normed space. B_{X^*} is weak*-compact so Krein-Milman implies $B_{X^*} = \overline{\text{conv}(\text{ext}(B_{X^*}))}^{\text{weak}^*}$.

65 (examples of extreme points) ex. If K is compact and Hausdorff, then $f \in \text{ext}(B_{C(K)}) \Leftrightarrow \|f\| = 1$ and $|f| \equiv 1$.

ex. see examples from HW

66 (extreme points of $C(K)$) Proposition: $\overline{\text{conv}(\text{ext}(B_{C_{\mathbb{C}}(K)}))} = B_{C_{\mathbb{C}}(K)} \Leftrightarrow K$ is Boolean.

Theorem: If K is compact and Hausdorff, then $B_{C_{\mathbb{C}}(K)} = \overline{\text{conv}(\text{ext}(B_{C_{\mathbb{C}}(K)}))}$.

Proposition: If K is compact and Hausdorff, then $\text{ext}(B_{C(K)^*}) = \{\alpha\delta_k \mid k \in K, |\alpha| = 1\}$.

67 (Banach-Stone) Suppose K_1, K_2 are compact Hausdorff. Then $C(K_1)$ is isometrically isomorphic to $C(K_2)$ if and only if K_1 is homeomorphic to K_2.

68 (Milman) If X is Hausdorff LCTVS and $M \subseteq X$ is compact with $C = \overline{\text{conv}(M)}$ compact. Then $\text{ext}(C) \subseteq M$.

69 (Kakatani fixed point theorem) We say T is an affine transformation if $T(\alpha x + (1-\alpha)y) = \alpha Tx + (1-\alpha)Ty$ for $0 \leq \alpha \leq 1$, $x, y \in K$.

G is equicontinuous if for all neighborhoods U of 0, there exists a neighborhood V of 0 such that for $x, y \in K$, if $x - y \in V$ then for all $T \in G$, $Tx - Ty \in U$.

We call p a fixed point of G if $G(p) = \{Tp \mid T \in G\} = \{p\}$.

Theorem: Suppose X is a LCTVS and $K \subseteq X$ is convex compact, and G is an equicontinuous group (under composition) of affine transformations on K. Then G has a fixed point.

70 (Haar measure) If G is a group and $\mathcal{T}$ is a topology, then $(G, \mathcal{T})$ is a topological group if $G \times G \to G$, $(g_1, g_2) \mapsto g_1 g_2^{-1}$ is continuous.

Note that this implies that $(g_1, g_2) \mapsto g_1 g_2$ and $g \mapsto g^{-1}$ are continuous.

Theorem: If $(G, \mathcal{T})$ is locally compact, then there exists a Biare measure μ on G such that

1. $\mu(K) < \infty$ for all $K \in \mathcal{B}a(G)$
2. $\mu(O) > 0$ is O is non-empty and open
3. $\mu(xE) = \mu(E)$ for every $E \in \mathcal{B}a(G)$ and $x \in G$ (that is, μ is left-invariant)

Moreover, this μ is unique and is also right invariant (i.e. $\mu(Ex) = \mu(E)$)

If $(G, \mathcal{T})$ is compact, then we can also get $\mu(G) = 1$.

71 (convex hull of compact dudes) **Theorem:** If X is a Banach space and $C \subseteq X$ is weakly compact, then $\overline{\text{conv}(C)}$ is weakly compact.

Theorem: If X is a Banach space and $C \subseteq X$ is closed, TFAE:

1. C is compact
2. there exists (x_n) in X such that $\|x_n\| \to 0$ and $C \subseteq \overline{\text{conv}(x_n)}$